

# SOME CHARACTERIZATIONS OF THE QUASI-SUM PRODUCTION MODELS WITH PROPORTIONAL MARGINAL RATE OF SUBSTITUTION

ALINA DANIELA VÎLCU, GABRIEL EDUARD VÎLCU

**ABSTRACT.** In this note we classify quasi-sum production functions with constant elasticity of production with respect to any factor of production and with proportional marginal rate of substitution.

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The notion of *production function* is a key concept in both macroeconomics and microeconomics, being used in the mathematical modeling of the relationship between the output of a firm, an industry, or an entire economy, and the inputs that have been used in obtaining it. Generally, production function is a twice differentiable mapping  $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ ,  $f = f(x_1, \dots, x_n)$ , where  $f$  is the quantity of output,  $n$  is the number of the inputs and  $x_1, \dots, x_n$  are the factor inputs. A production function  $f$  is called *quasi-sum* [3, 5] if there are strict monotone functions  $G, h_1, \dots, h_n$  with  $G' > 0$  such that

$$(1) \quad f(x) = G(h_1(x_1) + \dots + h_n(x_n)),$$

where  $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$ . We note that these functions are of great interest because they appear as solutions of the general bisymmetry equation, being related to the problem of consistent aggregation [1].

Among the family of production functions, the most famous is the so-called Cobb-Douglas production function. A generalized Cobb-Douglas production function depending on  $n$ -inputs is given by

$$(2) \quad f(x_1, \dots, x_n) = A \cdot \prod_{i=1}^n x_i^{\alpha_i},$$

where  $A, \alpha_1, \dots, \alpha_n > 0$ . We recall that a production function of the form  $f(x) = G(h(x_1, \dots, x_n))$ , where  $G$  is a strictly increasing function and  $h$  is a homogeneous function of any given degree  $p$ , is said to be a *homothetic* production function [7]. It is easy to see that a production function  $f$  can be identified with the graph of  $f$ , *i.e.* the nonparametric hypersurface of  $\mathbb{E}^{n+1}$  defined by

$$(3) \quad L(x_1, \dots, x_n) = (x_1, \dots, x_n, f(x_1, \dots, x_n))$$

and called the *production hypersurface* of  $f$  (see [9, 11]). Motivated by some recent classification results concerning production hypersurfaces [2, 5, 7, 8, 12], in the present work we classify quasi-sum production functions with proportional marginal

rate of substitution and investigate the existence of such production models whose production hypersurfaces have null Gauss-Kronecker curvature or null mean curvature. We recall that, if  $f$  is a production function with  $n$  inputs  $x_1, x_2, \dots, x_n$ ,  $n \geq 2$ , then the *elasticity of production* with respect to a certain factor of production  $x_i$  is defined as

$$(4) \quad E_{x_i} = \frac{x_i}{f} f_{x_i}$$

and the *marginal rate of technical substitution* of input  $x_j$  for input  $x_i$  is given by

$$(5) \quad \text{MRS}_{ij} = \frac{f_{x_j}}{f_{x_i}},$$

where the subscripts denote partial derivatives of the function  $f$  with respect to the corresponding variables. A production function satisfies the *proportional marginal rate of substitution property* if

$$(6) \quad \text{MRS}_{ij} = \frac{x_i}{x_j}, \text{ for all } 1 \leq i \neq j \leq n.$$

In the last section of the paper we will prove the following theorem which generalize the results from [10].

**Theorem 0.1.** *Let  $f$  be a quasi-sum production function given by (1). Then:*

- i. *The elasticity of production is a constant  $k_i$  with respect to a certain factor of production  $x_i$  if and only if  $f$  reduces to*

$$(7) \quad f(x_1, \dots, x_n) = A \cdot x_i^{k_i} \cdot \exp \left( D \sum_{j \neq i} h_j(x_j) \right),$$

*where  $A$  and  $D$  are positive constants.*

- ii. *The elasticity of production is a constant  $k_i$  with respect to all factors of production  $x_i$ ,  $i = 1, \dots, n$ , if and only if  $f$  reduces to the generalized Cobb-Douglas production function given by (2).*
- iii. *The production function satisfies the proportional marginal rate of substitution property if and only if it reduces to the homothetic generalized Cobb-Douglas production function given by*

$$(8) \quad f(x_1, \dots, x_n) = F \left( \prod_{i=1}^n x_i^k \right),$$

*where  $k$  is a nonzero real number.*

- iv. *If the production function satisfies the proportional marginal rate of substitution property, then:*
  - iv<sub>1</sub>. *The production hypersurface has vanishing Gauss-Kronecker curvature if and only if, up to a suitable translation,  $f$  reduces to the following generalized Cobb-Douglas production function with constant return to scale:*

$$(9) \quad f(x_1, \dots, x_n) = A \cdot \prod_{i=1}^n x_i^{\frac{1}{n}}.$$

- iv<sub>2</sub>. *The production hypersurface cannot be minimal.*

- iv<sub>3</sub>. *The production hypersurface has vanishing sectional curvature if and only if, up to a suitable translation,  $f$  reduces to the following generalized Cobb-Douglas production function:*

$$(10) \quad f(x_1, \dots, x_n) = A \cdot \prod_{i=1}^n \sqrt{x_i}.$$

## 1. PRELIMINARIES ON THE GEOMETRY OF HYPERSURFACES

For general references on the geometry of hypersurfaces, we refer to [4].

If  $M$  is a hypersurface of the Euclidean space  $\mathbb{E}^{n+1}$ , then it is known that the *Gauss map*  $\nu : M \rightarrow S^n$  maps  $M$  to the unit hypersphere  $S^n$  of  $\mathbb{E}^{n+1}$ . With the help of the differential  $d\nu$  of  $\nu$  it can be defined a linear operator on the tangent space  $T_p M$ , denoted by  $S_p$  and known as the *shape operator*, by  $g(S_p v, w) = g(d\nu(v), w)$ , for  $v, w \in T_p M$ , where  $g$  is the metric tensor on  $M$  induced from the Euclidean metric on  $\mathbb{E}^{n+1}$ . The eigenvalues of the shape operator are called *principal curvatures*. The determinant of the shape operator  $S_p$ , denoted by  $K(p)$ , is called the *Gauss-Kronecker curvature*. When  $n = 2$ , the Gauss-Kronecker curvature is simply called the *Gauss curvature*, which is intrinsic due to famous Gauss's Theorema Egregium. The trace of the shape operator  $S_p$  is called the *mean curvature* of the hypersurfaces. In contrast to the Gauss-Kronecker curvature, the mean curvature is extrinsic, which depends on the immersion of the hypersurface. A hypersurface is said to be *minimal* if its mean curvature vanishes identically. We recall now the following Lemma which will be used in the proof of Theorem 0.1.

**Lemma 1.1.** [4] *For the production hypersurface defined by (3) and  $w = \sqrt{1 + \sum_{i=1}^n f_i^2}$ ,*

*we have:*

- i. *The Gauss-Kronecker curvature  $K$  is given by*

$$(11) \quad K = \frac{\det(f_{x_i x_j})}{w^{n+2}}.$$

- ii. *The mean curvature  $H$  is given by*

$$(12) \quad H = \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \frac{f_{x_i}}{w} \right).$$

- iii. *The sectional curvature  $K_{ij}$  of the plane section spanned by  $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}$  is*

$$(13) \quad K_{ij} = \frac{f_{x_i x_i} f_{x_j x_j} - f_{x_i x_j}^2}{w^2 (1 + f_{x_i}^2 + f_{x_j}^2)}.$$

## 2. PROOF OF THEOREM 0.1

Let  $f$  be a quasi-sum production function given by (1). Then we have

$$(14) \quad f_{x_i}(x) = G'(u) h'_i(x_i)$$

with  $u = h_1(x_1) + \dots + h_n(x_n)$  and from (14) we derive

$$(15) \quad f_{x_i x_i} = G''(h'_i)^2 + G' h''_i, \quad i = 1, \dots, n,$$

$$(16) \quad f_{x_i x_j} = G'' h'_i h'_j, \quad i \neq j.$$

i. We first prove the left-to-right implication. If the elasticity of production is a constant  $k_i$  with respect to a certain factor of production  $x_i$ , then from (4) we obtain

$$(17) \quad f_{x_i} = k_i \frac{f}{x_i}.$$

Using now (1) and (14) in (17) we get

$$(18) \quad \frac{G'}{G} = k_i \frac{1}{x_i h'_i}.$$

By taking the partial derivative of (18) with respect to  $x_j$ ,  $j \neq i$ , we obtain

$$h'_j \frac{G''G - (G')^2}{G^2} = 0.$$

Now, taking into account that  $h_j$  is a strict monotone function, we find

$$(19) \quad G(u) = C \cdot e^{Du},$$

for some positive constants  $C$  and  $D$ . Hence from (18) and (19) we obtain

$$(20) \quad h_i(x_i) = \frac{k_i}{D} \ln x_i + A_i,$$

where  $A_i$  is a real constant. Finally, combining (1), (19) and (20) we get a function of the form (7), where  $A = Ce^{D \cdot A_i}$ . The converse can be verified easily by direct computation.

ii. The assertion is an immediate consequence of i.

iii. Assume first that  $f$  satisfies the proportional marginal rate of substitution property. Then from (5), (6) and (14) we derive  $x_i h'_i = x_j h'_j$ ,  $\forall i \neq j$ . Hence we conclude that there exists a nonzero real number  $k$  such that:  $x_i h'_i = k$ ,  $i = 1, \dots, n$ , and therefore we obtain

$$(21) \quad h_i(x_i) = k \ln x_i + C_i, \quad i = 1, \dots, n,$$

for some real constants  $C_1, \dots, C_n$ . Now, from (1) and (21) we derive

$$f(x) = G \left( k \sum_{i=1}^n \ln x_i + \overline{A} \right),$$

where  $\overline{A} = \sum_{i=1}^n C_i$  and hence we find

$$(22) \quad f(x) = (G \circ \ln) \left( A \cdot \prod_{i=1}^n x_i^k \right),$$

where  $A = e^{\overline{A}}$ . Therefore we get a production function of the form (8), where  $F(u) = (G \circ \ln)(A \cdot u)$ .

The converse is easy to verify.

iv<sub>1</sub>. We first prove the left-to-right implication. If the production hypersurface has null Gauss-Kronecker curvature, then from (11) we get

$$(23) \quad \det(f_{x_i x_j}) = 0.$$

On the other hand, the determinant of the Hessian matrix of  $f$  is given by [6]

$$(24) \quad \det(f_{x_i x_j}) = (G')^n \prod_{i=1}^n h_i'' + (G')^{n-1} G'' \sum_{i=1}^n h_1'' \cdots h_{i-1}'' (h_i')^2 h_{i+1}'' \cdots h_n''.$$

By using (21), (23) and (24), we obtain

$$(-1)^n (G')^{n-1} k^n (G' - knG'') = 0.$$

But  $G' > 0$  and  $k \neq 0$  and hence we derive

$$(25) \quad \frac{G''}{G'} = \frac{1}{kn}.$$

After solving (25) we find

$$(26) \quad G(u) = Cnk e^{\frac{u}{nk}} + D$$

for some constants  $C, D$  with  $C > 0$ . Combining (22) and (26), after a suitable translation, we conclude that the function  $f$  reduces to the form (9). The converse follows easily by direct computation.

iv<sub>2</sub>. Let us assume that the production hypersurface is minimal. Then we have  $H = 0$  and from (12) we derive

$$\sum_{i=1}^n f_{x_i x_i} \left( 1 + \sum_{i=1}^n f_{x_i}^2 \right) - \sum_{i,j=1}^n f_{x_i} f_{x_j} f_{x_i x_j} = 0$$

which reduces to

$$(27) \quad \sum_{i=1}^n f_{x_i x_i} + \sum_{i \neq j} (f_{x_i}^2 f_{x_j x_j} - f_{x_i} f_{x_j} f_{x_i x_j}) = 0.$$

By introducing (14), (15) and (16) in (27), we get

$$(28) \quad G'' \sum_{i=1}^n (h_i')^2 + G' \sum_{i=1}^n h_i'' + (G')^3 \sum_{i \neq j} (h_i')^2 h_j'' = 0.$$

By using now (21) in (28) and taking into account that  $k \neq 0$ , we obtain

$$(29) \quad (kG'' - G') \sum_{i=1}^n \frac{1}{x_i^2} - k^2 (G')^3 \sum_{i \neq j} \frac{1}{x_i^2 x_j^2} = 0.$$

But the only solution of the equation (29) is  $G(u) = \text{constant}$ , which is a contradiction because  $G' > 0$ . Hence the production hypersurface cannot be minimal.

iv<sub>3</sub>. Assume first that the production hypersurface has  $K_{ij} = 0$ . Then from (13) we get

$$(30) \quad f_{x_i x_i} f_{x_j x_j} - f_{x_i x_j}^2 = 0.$$

By introducing (14), (15) and (16) in (30), since  $G' \neq 0$ , we obtain

$$(31) \quad [(h_i')^2 h_j'' + (h_j')^2 h_i''] G'' + h_i'' h_j'' G' = 0.$$

By using now (21) in (31) and taking into account that  $k \neq 0$ , we obtain

$$(32) \quad \frac{G''}{G'} = \frac{1}{2k}.$$

After solving (32) we get

$$(33) \quad G(u) = 2kC e^{\frac{u}{2k}} + D$$

for some constants  $C, D$  with  $C > 0$ . Finally, combining (22) and (33), after a suitable translation, we conclude that the function  $f$  reduces to the Cobb-Douglas production function given by (10). The converse is easy to verify by direct computation.

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#### REFERENCES

- [1] J. Aczél, G. Maksa, *Solution of the rectangular  $m \times n$  generalized bisymmetry equation and of the problem of consistent aggregation*, J. Math. Anal. Appl. 203 (1996) 104-126.
- [2] M.E. Aydin, M. Ergüt, *A classification of homothetical hypersurfaces in Euclidean spaces via Allen determinants and its applications*, Appl. Sci. 17 (2015) 1-8.
- [3] M.E. Aydin, A. Mihai, *Classification of quasi-sum production functions with Allen determinants*, Filomat 29(6) (2015) 1351-1359.
- [4] B.-Y. Chen, *Pseudo-Riemannian geometry,  $\delta$ -invariants and applications*, World Scientific, Hackensack, NJ (2011).
- [5] B.-Y. Chen, *On some geometric properties of quasi-sum production models*, J. Math. Anal. Appl. 392(2) (2012) 192-199.
- [6] B.-Y. Chen, *An explicit formula of Hessian determinants of composite functions and its applications*, Kragujevac J. Math. 36 (2012) 1-14.
- [7] B.-Y. Chen, *Solutions to homogeneous Monge-Ampère equations of homothetic functions and their applications to production models in economics*, J. Math. Anal. Appl. 411 (2014) 223-229.
- [8] Y. Fu, W.G. Wang, *Geometric characterizations of quasi-product production models in economics*, Filomat (2015), in press.
- [9] A.D. Vilcu, G.E. Vilcu, *On some geometric properties of the generalized CES production functions*, Appl. Math. Comput. 218(1) (2011) 124-129.
- [10] A.D. Vilcu, G.E. Vilcu, *On homogeneous production functions with proportional marginal rate of substitution*, Math. Probl. Eng. 2013 (2013), Article ID 732643, 5 pages.
- [11] G.E. Vilcu, *A geometric perspective on the generalized Cobb-Douglas production functions*, Appl. Math. Lett. 24(5) (2011) 777-783.
- [12] X. Wang, Y. Fu, *Some characterizations of the Cobb-Douglas and CES production functions in microeconomics*, Abstr. Appl. Anal. 2013 (2013), Article ID 761832, 6 pages.

Alina Daniela VÎLCU

Petroleum-Gas University of Ploiești,  
Department of Computer Science, Information Technology, Mathematics and Physics,  
Bulevardul București, Nr. 39, Ploiești 100680-ROMANIA  
e-mail: daniela.vilcu@upg-ploiesti.ro

Gabriel Eduard VÎLCU<sup>1,2</sup>

<sup>1</sup>University of Bucharest, Faculty of Mathematics and Computer Science,  
Research Center in Geometry, Topology and Algebra,  
Str. Academiei, Nr. 14, Sector 1,  
București 70109-ROMANIA  
e-mail: gvilcu@gta.math.unibuc.ro

<sup>2</sup>Petroleum-Gas University of Ploiești,  
Department of Mathematical Modelling, Economic Analysis and Statistics,  
Bulevardul București, Nr. 39, Ploiești 100680-ROMANIA  
e-mail: gvilcu@upg-ploiesti.ro